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Two players play a game. Each player, in turn, has to name a positive integer that is less than the previous number but at least half the previous number. The player who names the number 1 loses. If the first player starts by naming 2003 and after that both players play with the best strategy, who wins? Describe the strategy and prove it works.

Solution by William Chau, East Brunswick, New Jersey.

Let a_k and b_k denote the k-th numbers named by the first and the second players, respectively. Given that $a_1 = 2003$ and both players play with the best strategy, it can be shown by induction that, for $1 \le k \le 10$, b_k exists (the second player names at least k numbers before the game ends) and $b_k = 3 \cdot 2^{10-k} - 1$ is a possible pick for any first player's strategy. It follows that $b_{10} = 2$ if $b_k = 3 \cdot 2^{10-k} - 1$ for $1 \le k \le 10$. The rule of the game forces $b_{10}/2 = 1 \le a_{11} < b_{10} = 2$, so $a_{11} = 1$ and the second player wins the game.

Since $1 < a_1/2 \le 3 \cdot 2^9 - 1 = 1535 < a_1$, the game continues to the second player's turn and $b_1 = 3 \cdot 2^{10-1} - 1$ is a possible pick. Assume that $b_k = 3 \cdot 2^{10-k} - 1$ for $1 \le k \le 9$. Since $b_k \ge 3 \cdot 2^{10-9} - 1 = 5 > 1$, the game continues to the first player's turn. The rule of the game requires that $a_{k+1} < b_k = 3 \cdot 2^{10-k} - 1$ and $a_{k+1} \ge b_k/2 > \left(3 \cdot 2^{10-k} - 2\right)/2 = 3 \cdot 2^{9-k} - 1$. Therefore $3 \cdot 2^{9-k} - 1 < a_{k+1} < 3 \cdot 2^{10-k} - 1$ for any first player's strategy. Since $a_{k+1} > 3 \cdot 2^{9-9} - 1 = 2 > 1$, the game continues to the second player's turn. Similarly, $a_{k+1}/2 \le \left(3 \cdot 2^{10-k} - 2\right)/2 = 3 \cdot 2^{9-k} - 1 < a_{k+1}$ gives the possibility of the pick of $b_{k+1} = 3 \cdot 2^{10-(k+1)} - 1$. Hence the proposition of the induction is established.